Journal of Statistical Physics, Vol. 124, No. 1, July 2006 (© 2006)
DOI: 10.1007/s10955-006-9140-9

Suspension Flows Over Countable Markov Shifts

Luis Barreira[1] and Godofredo Iommi[2]

Received February 14, 2006; accepted May 31, 2006
Published Online: July 12, 2006

We introduce the notion of topological pressure for suspension flows over *countable* Markov shifts, and we develop the associated thermodynamic formalism. In particular, we establish a variational principle for the topological pressure, and an approximation property in terms of the pressure on compact invariant sets. As an application we present a multifractal analysis for the entropy spectrum of Birkhoff averages for suspension flows over countable Markov shifts. The domain of the spectrum may be unbounded and the spectrum may not be analytic. We provide explicit examples where this happens. We also discuss the existence of full measures on the level sets of the multifractal decomposition.

KEY WORDS: countable Markov shifts, suspension flows.
2000 Mathematics Subject Classification Primary: 37D35

1. INTRODUCTION

Our work is devoted to the study of suspension flows over countable Markov shifts. In order to provide motivation, we first recall the notion of suspension semiflow and its relation to the study of axiom A flows on compact manifolds. It was shown by Bowen(5) and Ratner(14) that axiom A flows on compact manifolds can be modeled by suspension flows over Markov shifts with a finite alphabet, as a consequence of the existence of the so-called Markov systems. Therefore, a detailed study of suspension flows may provide important information about the dynamics of Axiom A flows.

An important assumption in these works is that the flow is defined on a compact manifold. It was conjectured by Sinai that geodesic flows on noncompact

[1] Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal; e-mail: barreira@math.ist.utl.pt.
[2] Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal; e-mail: giommi@math.ist.ntl.pt.

0022-4715/06/0700-0207/0 © 2006 Springer Science+Business Media, Inc.

manifolds of negative curvature and finite volume have associated a symbolic dynamics with countably many symbols. A similar situation may occur in the context of nonuniformly hyperbolic dynamics. Thus, understanding the dynamics of suspension flows over countable Markov shifts should provide important information, for example for geodesic flows on noncompact manifolds of negative curvature. Progress in this direction was achieved by Gurevich and Katok,[9] who proved that the geodesic flow on the modular surface can be modeled by a suspension flow over a countable alphabet.

In this paper, motivated by these considerations, we study the ergodic properties of suspension flows over countable Markov shifts. In particular, we develop a thermodynamic formalism for these flows. We also use it to obtain a multifractal analysis of Birkhoff averages.

In the case of Markov shifts with a finite alphabet there is a canonical identification between the invariant probability measures for the suspension flow and the shift map (in the base of the flow); see Sec. 2.1 for details. This relation was exploited by Bowen and Ruelle (see Ref.6) to prove that the properties of the topological pressure for the suspension flow are similar to those of the topological pressure for the Markov shift. In the case of countable Markov shifts, a bijection between invariant probability measures for the suspension flow and the shift on the base may not exist, because the height function may not be integrable with respect to some measures invariant under the shift. Therefore, the thermodynamic formalism for the suspension flow need not to be related to the one on the base. Thus, one needs to introduce a new topological pressure.

We propose a notion of topological pressure based on the Gurevich pressure for countable Markov shifts (see the work by Sarig[15] and Sec. 2.2 for definitions), and on the relation used by Bowen and Ruelle to translate problems for the flow into problems for the Markov shift. Our notion extends the notion of entropy for suspension flows over countable Markov shifts introduced by Savchenko (see Ref.17), although he considered height functions depending only on the first coordinate. We establish several properties of the pressure, namely a variational principle (see Theorem 2) and an approximation property in terms of the pressure on compact invariant sets (see Theorem 1). Examples are provided in Sec. 4.

As an application of the above construction, we present a multifractal analysis for the entropy spectrum of Birkhoff averages for suspension flows over countable Markov chains (see Theorem 10). For this we also need to introduce a notion of topological entropy on noncompact sets that in general are not subsets of compact invariant sets (contrarily to what happens in Bowen's classical notion of topological entropy on noncompact sets). We note that our notion of topological entropy is an extension both of Bowen's notion and of the topological entropy obtained from our notion of topological pressure simply by considering the zero potential.

Our work on the multifractal analysis extends results obtained by Barreira and Saussol[2,3] and Pesin and Sadovskaya[13] for suspension flows over finite Markov shifts as well as results by Iommi[10] for countable Markov shifts. We recall that for suspension flows over finite Markov shifts the entropy spectrum is analytic and has bounded domain. In strong contrast, in the case of countable Markov shifts the spectrum may have unbounded domain and need not be analytic. We give explicit examples where this happens (see Sec. 5.4). We also provide a classification of the spectra for which there exist full measures on the level sets of the multifractal decomposition (see Theorem 12).

2. PRELIMINARIES

2.1. Suspension Flows and Invariant Measures

Let $\sigma : \Sigma \to \Sigma$ be a one-sided Markov shift with a countable alphabet S. This means that there exists a matrix $(t_{ij})_{S\times S}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma = \{x \in S^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \quad \text{for every } i \in \mathbb{N}_0\},$$

and the shift map is defined by $\sigma(x_0 x_1 \cdots) = (x_1 x_2 \cdots)$. Sometimes we simply say that σ is a *countable Markov shift*. Let now $\tau : \Sigma \to \mathbb{R}^+$ be a continuous function and consider the space

$$Y = \{(x, t) \in \Sigma \times \mathbb{R} : 0 \leq t \leq \tau(x)\}, \tag{1}$$

with the points $(x, \tau(x))$ and $(\sigma(x), 0)$ identified for each $x \in \Sigma$. The *suspension semiflow* over σ with *height function* τ is the semiflow $\Phi = (\varphi_t)_{t\geq 0}$ on Y defined by

$$\varphi_t(x, s) = (x, s + t) \text{ whenever } s + t \in [0, \tau(x)].$$

In the case of two-sided Markov shifts we can define a suspension flow $(\varphi_t)_{t\in\mathbb{R}}$ in a similar manner.

We denote by $\mathcal{M}_\Phi$ the space of Φ-invariant probability measures on Y. Recall that a measure μ on Y is Φ-*invariant* if $\mu(\varphi_t^{-1} A) = \mu(A)$ for every $t \geq 0$ and every measurable set $A \subset Y$. We also consider the space $\mathcal{M}_\sigma$ of σ-invariant probability measures on Σ. Given a continuous function $\phi : \Sigma \to \mathbb{R}$ we consider the set

$$\mathcal{M}_\sigma(\phi) := \left\{ \nu \in \mathcal{M}_\sigma : -\int_\Sigma \phi d\nu < \infty \right\}.$$

One can easily verify that if ν is a σ-invariant measure on Σ, possibly infinite, such that $\int_\Sigma \tau d\nu < \infty$ and m is the Lebesgue measure on $\mathbb{R}$, then the finite measure induced on Y by the product measure $\nu \times m$ is Φ-invariant. Moreover, when τ is bounded away from zero there is a canonical identification between $\mathcal{M}_\Phi$ and $\mathcal{M}_\sigma(-\tau)$. Namely, the map $R : \mathcal{M}_\sigma(-\tau) \to \mathcal{M}_\Phi$ defined by

$$R(\nu) = (\nu \times m)|_Y / (\nu \times m)(Y) \tag{2}$$

is a bijection. We note that if $\sigma : \Sigma \to \Sigma$ is a Markov shift with a finite alphabet then τ is bounded and bounded away from zero (since Σ is compact). In particular $\int_\Sigma \tau d\nu < \infty$ for every $\nu \in \mathcal{M}_\sigma$. Therefore, in this case $\mathcal{M}_\sigma(-\tau) = \mathcal{M}_\sigma$ and the map R is a bijection between $\mathcal{M}_\Phi$ and $\mathcal{M}_\sigma$. We emphasize that in the general case of countable Markov shifts the map R need not be a bijection and this causes additional difficulties.

Given a continuous function $g : Y \to \mathbb{R}$ we define a function $\Delta_g : \Sigma \to \mathbb{R}$ by

$$\Delta_g(x) = \int_0^{\tau(x)} g(x, t)dt.$$

Clearly, Δ_g is also continuous. We have

$$\int_Y g\, dR(\nu) = \frac{\int_\Sigma \Delta_g d\nu}{\int_\Sigma \tau d\nu}. \tag{3}$$

When σ is a Markov shift with a finite alphabet and $g : Y \to \mathbb{R}$ is Hölder continuous, it was shown by Bowen and Ruelle in Ref.6 (see also Ref.12) that the topological pressure of g with respect to the semiflow Φ, denoted by $P_\Phi(g)$, is related to the topological pressure P with respect to the shift by the formula

$$P(\Delta_g - P_\Phi(g)\tau) = 0.$$

This relation relies on the fact that in this case R is a bijection and on the variational principle for the topological pressure (together with (3)).

2.2. Thermodynamic Formalism for Countable Markov Shifts

We recall here some notions from the thermodynamic formalism for countable shifts. We refer to Refs. 15 and 16 for more details.

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing countable Markov shift. This means that $\sigma|\Sigma$ is a topologically mixing dynamical system when Σ is equipped with the topology generated by the cylinder sets

$$C_{a_0 \cdots a_n} = \{x \in \Sigma : x_i = a_i \text{ for } i = 0, \dots, n\}. \tag{4}$$

Given a function $\phi : \Sigma \to \mathbb{R}$ we define

$$V_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma,\ x_i = y_i \text{ for } i = 0, \dots, n-1\},$$

where $x = (x_0 x_1 \cdots)$ and $y = (y_0 y_1 \cdots)$. We say that ϕ is *locally Hölder* if there exist constants $B > 0$ and $\theta \in (0, 1)$ such that $V_n(\phi) \leq B\theta^n$ for all $n \in \mathbb{N}$. Note that since nothing is required for $n = 0$ a locally Hölder function is not necessarily bounded.

We now introduce the notion of (topological) pressure for a countable Markov shift. Fix a symbol i_0 in the alphabet S and let $\phi : \Sigma \to \mathbb{R}$ be a locally Hölder function. The so-called *Gurevich pressure* of ϕ was introduced by Sarig in Ref.15 as

$$P_\sigma(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x:\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(\sigma^i x)\right) \chi_{C_{i_0}}(x),$$

where $\chi_{C_{i_0}}(x)$ is the characteristic function of the cylinder $C_{i_0} \subset \Sigma$ (see (4)). Since σ is topologically mixing one can show that $P_\sigma(\phi)$ does not depend on i_0. Furthermore, the following properties hold:

1. (approximation property) if

$$\mathcal{K} := \{K \subset \Sigma : K \neq \emptyset \text{ compact and } \sigma\text{-invariant}\},$$

then

$$P_\sigma(\phi) = \sup\{P_{\sigma|K}(\phi) : \ K \in \mathcal{K}\}, \tag{5}$$

where $P_{\sigma|K}$ is the classical topological pressure on K;

2. (variational principle) if $\sup_\Sigma \phi < \infty$ then

$$P_\sigma(\phi) = \sup\left\{h_\mu(\sigma) + \int_\Sigma \phi\, d\mu : \mu \in \mathcal{M}_\sigma(\phi)\right\}. \tag{6}$$

There is a certain class of countable Markov shifts for which the thermodynamic formalism is particularly similar to the one for Markov shifts with a finite alphabet. We say that Σ satisfies the *big images and preimages property (BIP property)* if there exist $b_1, b_2, \ldots, b_n \in S$ such that for every

$$a \in S \text{ there exist } i, j \in S \text{ with } t_{b_i a} t_{a b_j} = 1.$$

Here t_{ij} are the entries of the transition matrix of Σ. We say that $\mu \in \mathcal{M}_\sigma$ is a *Gibbs measure* for the function $\phi : \Sigma \to \mathbb{R}$ if for some constants $P \in \mathbb{R}$ and $C > 0$, and every $n \in \mathbb{N}$ and $x \in C_{a_0\cdots a_n}$ we have

$$\frac{1}{C} \leq \frac{\mu(C_{a_0\cdots a_n})}{\exp(-nP + \sum_{i=0}^{n} \phi(\sigma^k x))} \leq C.$$

It was proved by Sarig in Ref.16 that a locally Hölder function ϕ with finite Gurevich pressure has an invariant Gibbs measure if and only if Σ satisfies the BIP property. The "if" part also follows from work of Mauldin and Urbański in Ref.11 (see Ref.16 for details). Moreover, if Σ satisfies the BIP property then the function $t \mapsto P_\sigma(t\phi)$ is real analytic for $t > 1$.

2.3. The Bowen-Walters Distance

Bowen's notion of topological entropy on an arbitrary subset of a compact invariant set requires a distance on the ambient space. In the case of suspension flows one uses the distance introduced by Bowen and Walters in Ref.7, that we briefly recall in this section. Bowen's notion of topological entropy will be used in Sec. 5.1 (see (16)).

Let again $\sigma : \Sigma \to \Sigma$ be a countable Markov shift. Given $\theta \in (0, 1)$, we define a distance on Σ by

$$d_\Sigma(x, y) = \theta^{\sup\{n \in \mathbb{N}_0 : x_n \neq y_n\}}.$$

When $\tau = 1$ on Σ, we introduce the so-called Bowen-Walters distance d_1 on Y in the following manner. Given $x, y \in \Sigma$ and $t \in [0, 1]$ we define the length of the *horizontal segment* $[(x, t), (y, t)]$ by

$$\rho_h((x, t), (y, t)) = (1 - t) d_\Sigma(x, y) + t d_\Sigma(\sigma x, \sigma y).$$

Note that

$$\rho_h((x, 0), (y, 0)) = d_\Sigma(x, y) \quad \text{and} \quad \rho_h((x, 1), (y, 1)) = d_\Sigma(\sigma x, \sigma y).$$

Furthermore, given $(x, t), (y, s) \in Y$ on the same orbit we define the length of the *vertical segment* $[(x, t), (y, s)]$ by

$$\rho_v((x, t), (y, s)) = \inf\{|r| : \varphi_r(x, t) = (y, s) \text{ and } r \in \mathbb{R}\}.$$

Finally, given arbitrary points $(x, t), (y, s) \in Y$ the distance $d_1((x, t), (y, s))$ is defined as the infimum of the lengths of paths between (x, t) and (y, s) composed of a finite number of horizontal and vertical segments.

For an arbitrary height function τ, the Bowen-Walters distance d_Y on Y between the points $(x, t), (y, s) \in Y$ is defined by

$$d_Y((x, t), (y, s)) = d_1((x, t/\tau(x)), (y, s/\tau(s))).$$

3. TOPOLOGICAL PRESSURE FOR SUSPENSION SEMIFLOWS

3.1. Notion of Topological Pressure

We now start developing a thermodynamic formalism for suspension semiflows over *countable* Markov shifts. Consider the suspension semiflow Φ over the countable Markov shift σ with height function τ locally Hölder and bounded away from zero. For a continuous function $g : Y \to \mathbb{R}$ such that Δ_g is locally Hölder, we define the *topological pressure* of g with respect to Φ by

$$P_\Phi(g) := \inf\{t \in \mathbb{R} : P_\sigma(\Delta_g - t\tau) \leq 0\}$$

(with the convention that $P_\Phi(g) = \infty$ when the infimum is taken over the empty set). Here P_σ is the Gurevich pressure.

We also define the *topological entropy* of the suspension semiflow Φ by

$$h(\Phi) := P_\Phi(0) = \inf\{t \in \mathbb{R} : P_\sigma(-t\tau) \leq 0\}. \tag{7}$$

Savchenko[17] was the first to introduce the notion of entropy of a suspension flow over a *countable* Markov shift. More precisely, he considered the particular case of height functions depending only on the first coordinate, i.e., $\tau(x) = \tau(x_0)$, and he defined the topological entropy of Φ by

$$\bar{h}(\Phi) := \sup\{h_\mu(\varphi_1) : \mu \in \mathcal{M}_\Phi\}.$$

On the other hand, he did not assume τ to be bounded away from zero. The difficulty that arises from this is that the map R defined by (2) may not be a bijection (it may not be surjective, in which case certain measures in $\mathcal{M}_\Phi$ are of the form $\nu \times m$, where ν is an infinite σ-invariant measure on Σ and m is the Lebesgue measure). Savchenko proved in Theorem 2 of Ref. 17 that $\bar{h}(\Phi) = \inf\{t : P_\sigma(-t\tau) \le 0\}$. Therefore, when τ depends only on the first coordinate and is bounded away from zero, we have $h(\Phi) = \bar{h}(\Phi)$, i.e., the two notions coincide.

Example 1. Even when the Gurevich entropy of the Markov shift in the base is infinite the entropy of the suspension semiflow may be finite. For example, let $\sigma : \Sigma \to \Sigma$ be the full shift on the countable alphabet $S = \mathbb{N}$, and let $\tau : \Sigma \to \mathbb{R}^+$ be the height function defined by

$$\tau(x) = \log(x_0(x_0 + 1)), \quad \text{where } x = (x_0 x_1 \cdots). \tag{8}$$

In this case $h(\sigma) = \infty$. Since we are considering the full shift, a point $x \in C_{i_0}$ satisfies $\sigma^n = x$ if and only if it is obtained by repeating a finite sequence $i_0 j_1 \cdots j_{n-1}$, with $j_1, \ldots, j_{n-1} \in \mathbb{N}$. Therefore, for a function $\phi(x) = \phi(x_0)$ we have

$$\begin{aligned} P(-t\log\phi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \prod_{i=0}^{n-1} (\phi(\sigma^i x))^{-t} \chi_{C_{i_0}}(x) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{j_1 j_2, \ldots, j_{n-1} \in \mathbb{N}} \lambda_{i_0}^{-t} (\lambda_{j_1} \ldots \lambda_{j_{n-1}})^{-t} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \lambda_{i_0}^{-t} + \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i=1}^{\infty} (\lambda_i^{-t}) \right)^{n-1} \\ &= \log \sum_{i=1}^{\infty} \lambda_i^{-t}. \end{aligned} \tag{9}$$

In particular, for the function τ in (8) we obtain

$$P(-t\tau) = \log \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right)^t .$$

Hence, $h(\Phi) = 1 = \inf\{t : P_\sigma(-t\tau) \le 0\}$.

Example 2. The equation $P(\Delta_g - t\tau) = 0$ may have no root. Let again $\sigma : \Sigma \to \Sigma$ be the full shift on a countable alphabet. Let $\alpha(n) = 2n(\log 2n)^2$ and take $N > 0$ such that $\Sigma_{n>N}\alpha(n)^{-1} < 1$. We consider the height function $\tau : \Sigma \to \mathbb{R}^+$ defined by

$$\tau(x) = \log \alpha(x_0 + N), \quad \text{where } x = (x_0 x_1 \cdots).$$

Then

$$P_\sigma(-t\tau) = \begin{cases} \infty & \text{if } t < 1, \\ \text{negative} & \text{if } t \geq 1. \end{cases}$$

For $t \geq 1$ this follows from (9) and the choice of N. Hence, the topological entropy of the associated suspension semiflow Φ is

$$h(\Phi) = 1 = \inf\{t : P_\sigma(-t\tau) \leq 0\},$$

and $P_\sigma(-h(\Phi)\tau) < 0$.

3.2. Basic Properties of the Pressure

In the next statement the pressure is described as the supremum over the compact invariant sets.

Theorem 1. (Approximation property). *Let Φ be a suspension semiflow on Y over a countable Markov shift. If $g : Y \to \mathbb{R}$ is a continuous function such that Δ_g is locally Hölder and bounded above, then*

$$P_\Phi(g) = \sup\{P_{\Phi|K}(g) : K \subset Y \textit{ compact and } \Phi\textit{-invariant}\}.$$

Proof: By the classical theory, for any compact sets $K_1 \subset K_2 \subset Y$ we have

$$P_{\Phi|K_1}(g) \leq P_{\Phi|K_2}(g).$$

It follows from (5) that

$$\begin{aligned} P_\Phi(g) &= \inf\{t \in \mathbb{R} : \sup_{K \in \mathcal{K}} P_{\sigma|K}(\Delta_g - t\tau) \leq 0\} \\ &\geq \inf\{t \in \mathbb{R} : P_{\sigma|K}(\Delta_g - t\tau) \leq 0\} = P_{\Phi|K}(g) \qquad (10) \end{aligned}$$

for each compact set $K \subset \Sigma$. On the other hand, by (5), the topological pressure of the Markov shift satisfies

$$P_\sigma(\Delta_g - t\tau) = \sup\{P_{\sigma|K}(\Delta_g - t\tau) : K \subset \Sigma \text{ compact and } \sigma\text{-invariant}\}. \quad (11)$$

Let $Y_K \subset Y$ be the compact and Φ-invariant set having for base the compact and σ-invariant set $K \subset \Sigma$. Let also $P_{\Phi|Y_K}(g)$ be the unique real number satisfying

$$P_{\sigma|K}(\Delta_g - P_{\Phi|Y_K}(g)\tau) = 0 \tag{12}$$

(compactness ensures that such a number exists). Note that $P_{\Phi|Y_K}(g)$ is indeed the topological pressure of g on Y_K.

Setting

$$A := \sup\{P_{\Phi|Y_K}(g) : K \subset Y \text{ compact and } \Phi\text{-invariant}\},$$

it follows from (10) that $A \leq P_\Phi(g)$. We claim that equality holds. Assume on the contrary that $A < P_\Phi(g)$ and let $s \in (A, P_\Phi(g))$ (we can assume that A is finite, since otherwise the result is immediate). Since τ is positive, the function $t \mapsto P_{\sigma|K}(\Delta_g - t\tau)$ is decreasing. Since $s > A$, it follows from (11) and (12) that $P_{\sigma|K}(\Delta_g - s\tau) \leq 0$ for every compact σ-invariant set $K \subset \Sigma$, and hence $P_\sigma(\Delta_g - s\tau) \leq 0$. On the other hand, since $s < P_\Phi(g)$ we have $P_\sigma(\Delta_g - s\tau) > 0$. This contradiction proves the result. □

By Theorem 1 the topological pressure P_Φ is a convex function of g, since it is the supremum of convex functions.

We now establish a variational principle for suspension semiflows over countable Markov shifts.

Theorem 2. (Variational principle). *Let Φ be a suspension semiflow on Y over a countable Markov shift. If $g : Y \to \mathbb{R}$ is a continuous function such that Δ_g is locally Hölder and bounded above, then*

$$P_\Phi(g) = \sup\left\{h_\mu(\Phi) + \int_Y g\,d\mu : \mu \in \mathcal{M}_\Phi \text{ and } -\int_Y g\,d\mu < \infty\right\}. \tag{13}$$

Proof: Set $\mathcal{N} = \mathcal{M}_\sigma(-\tau) \cap \mathcal{M}_\sigma(\Delta_g)$. By the variational principle for countable Markov shifts (see (6)), for each $t > P_\Phi(g)$ we have

$$\begin{aligned} 0 &\geq P_\sigma(\Delta_g - t\tau) \\ &\geq \sup\left\{h_\nu(\sigma) + \int_\Sigma \Delta_g d\nu - t\int_\Sigma \tau d\nu : \nu \in \mathcal{N}\right\}, \end{aligned}$$

since $\mathcal{N} \subset \mathcal{M}_\sigma(\Delta_g - t\tau)$. Therefore,

$$0 \geq \sup\left\{\int_\Sigma \tau d\nu \left(\frac{h_\nu(\sigma)}{\int_\Sigma \tau d\nu} + \frac{\int_\Sigma \Delta_g d\nu}{\int_\Sigma \tau d\nu} - t\right) : \nu \in \mathcal{N}\right\}. \tag{14}$$

For every $\nu \in \mathcal{N}$ we have

$$\int_Y g d(\nu \times m) < \infty \text{ if and only if } \int_\Sigma \Delta_g d\nu < \infty.$$

Since $\tau > 0$, by (3) and Abramov's formula, it follows from (14) that

$$0 \geq \sup \left\{ h_\mu(\Phi) + \int_Y g d\mu - t : \mu \in \mathcal{M}_\Phi \text{ and } -\int_Y g d\mu < \infty \right\}.$$

That is, $P \leq t$ where P is the supremum in (13). Hence, $P \leq P_\Phi(g)$.

For the reverse inequality, let $K \subset Y$ be a compact Φ-invariant set. Then

$$\sup \left\{ h_\mu(\Phi|K) + \int_K g d\mu : \mu \in \mathcal{M}_{\phi|K} \right\} \leq P.$$

By Theorem 1 we obtain $P_\Phi(g) \leq P$ and the proof is complete. □

Setting $g = 0$ in Theorems 1 and 2 we obtain the following.

Theorem 3. *Let* Φ *be a suspension semiflow on* Y *over a countable Markov shift. Then*

$$\begin{aligned} h(\Phi) &= \sup\{h(\Phi|K) : K \subset Y \; compact \; and \; \Phi\text{-}invariant\} \\ &= \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}_\Phi\}. \end{aligned}$$

Let now $g : Y \to \mathbb{R}$ be a continuous function such that Δ_g is locally Hölder. A measure $\mu \in \mathcal{M}_\Phi$ is called an *equilibrium measure* for g if

$$P_\Phi(g) = h_\mu(\Phi) + \int_Y g d\mu.$$

We will use the notation $u_g = \Delta_g - P_\Phi(g)\tau$.

Theorem 4. *Let* Φ *be a suspension semiflow on* Y *over a countable Markov shift, and let* $g : Y \to \mathbb{R}$ *be a continuous function such that* Δ_g *is locally Hölder and bounded above. Then the following properties are equivalent:*

1. *there is an equilibrium measure* $\mu_g \in \mathcal{M}_\Phi$ *for* g*;*
2. $P_\sigma(u_g) = 0$ *and there is an equilibrium measure* $\nu_g \in \mathcal{M}_\sigma(-\tau)$ *for* u_g*.*

Proof: By the definition of topological pressure $P_\sigma(u_g) \leq 0$. We assume first that $P_\sigma(u_g) < 0$. By (6), given $\nu \in \mathcal{M}_\sigma(-\tau)$ we have

$$h_\nu(\sigma) + \int_\Sigma \Delta_g d\nu - P_\Phi(g) \int_\Sigma \tau d\nu < 0$$

Since $\mathcal{M}_\sigma(-\tau)$ can be identified with $\mathcal{M}_\Phi$, using Abramov's formula we obtain that for every $\mu \in \mathcal{M}_\Phi$,

$$h_\mu(\Phi) + \int_Y g d\mu < P_\Phi(g),$$

and there are no equilibrium measures in this case.

Assume now that $P_\sigma(\mu_g) = 0$, and let $\nu_g \in \mathcal{M}_\sigma(-\tau)$ be an equilibrium measure for u_g. Then

$$P_\sigma(\mu_g) = h_{\nu_g}(\sigma) + \int_\Sigma u_g d\nu_g = 0.$$

Set $\mu_g = R(\nu_g)$. Since $\nu_g \in \mathcal{M}_\sigma(-\tau)$ we have $\int_\Sigma \tau d\nu_g < \infty$, and thus

$$P_\Phi(g) = \frac{h_{\nu_g}(\sigma)}{\int_\Sigma \tau d\nu_g} + \frac{\int_\Sigma \Delta_g d\nu_g}{\int_\Sigma \tau d\nu_g} = h_{\mu_g}(\Phi) + \int_Y g d\mu_g.$$

This shows that μ_g is an equilibrium measure for g. On the other hand, if we start with an equilibrium measure μ_g for g, then

$$P_\Phi(g) = h_{\mu_g}(\Phi) + \int_Y g d\mu_g.$$

The measure μ_g is obtained from a product measure $\nu_g \times m$ for some $\nu_g \in \mathcal{M}_\sigma(-\tau)$. Therefore, using Abramov's formula,

$$0 = P_\sigma(u_g) \geq h_{\nu_g}(\sigma) + \int_\Sigma u_g d\nu_g = 0.$$

In particular, ν_g is an equilibrium measure for u_g. This completes the proof. □

An equilibrium measure for the zero function $g \equiv 0$ is called a *measure of maximal entropy*. Theorem 4 implies that the following are equivalent:

1. there is a measure of maximal entropy in $\mathcal{M}_\Phi$;
2. $P_\sigma(-h(\Phi)\tau) = 0$ and $-h(\Phi)\tau$ has an equilibrium measure in $\mathcal{M}_\sigma(-\tau)$.

4. EXAMPLES

4.1. Bounded Height Function

When the height function is bounded, the properties of the pressure on the base (for the Markov shift) can be translated to the pressure for the flow. We note that when the height function τ is bounded, the map $R : \mathcal{M}_\sigma \to \mathcal{M}_\Phi$ in (2) is a bijection (since $\mathcal{M}_\sigma(-\tau) = \mathcal{M}_\sigma$).

Proposition 5. *If the height function is bounded, then either the equation $P_\sigma(\Delta_g - t\tau) = 0$ has a root or $P_\sigma(\Delta_g - t\tau) = \infty$ for every $t \in \mathbb{R}$.*

Proof: If $P_\sigma(\Delta_g) = \infty$ then $P_\Phi(g) = \infty$. Assume now that $P_\sigma(\Delta_g) < \infty$. Taking numbers $s, S > 0$ such that $s \le \tau \le S$, we obtain

$$-tS + P_\sigma(\Delta_g) \le P_\sigma(\Delta_g - t\tau) \le -ts + P_\sigma(\Delta_g).$$

Also, there exist numbers $t_s, t_S \in \mathbb{R}$ such that

$$0 \le -t_s s + P_\sigma(\Delta_g) < \infty \quad \text{and} \quad -\infty < -t_S S + P_\sigma(\Delta_g) \le 0.$$

The result follows from the continuity of the pressure. □

Proposition 6. (BIP shift). *Assume that σ satisfies the BIP property, and let $g : Y \to \mathbb{R}$ be a continuous function such that Δ_g is locally Hölder. If the height function is bounded, then the function $t \mapsto P_\Phi(tg)$, when finite, is real analytic.*

Proof: Recall that when σ satisfies the BIP property and Δ_g is locally Hölder, the function $t \mapsto P_\sigma(\Delta_g - t\tau)$, when finite, is real analytic. The result now follows from the implicit function theorem: $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$, and in order to verify the nondegeneracy condition note that

$$\frac{\partial}{\partial t} P_\sigma(\Delta_g - t\tau)\Big|_{t=s} = -\int_\Sigma \tau \, d\mu < 0\,,$$

where μ denotes the equilibrium measure of $\Delta_g - s\tau$. □

4.2. Bounded Potentials and BIP Shifts

We now assume that:

1. σ satisfies the BIP property;
2. $g : Y \to \mathbb{R}$ is bounded and Δ_g is locally Hölder;
3. there exists $t_c > 0$ such $P_\sigma(-t\tau) < \infty$ for every $t > t_c$,

$$\lim_{t \to t_c^+} P_\sigma(-t\tau) = \infty \quad \text{and} \quad \lim_{t \to \infty} P_\sigma(-t\tau) = -\infty. \tag{15}$$

Proposition 7. *The equation $P_\sigma(\Delta_g - t\tau) = 0$ has a root.*

Proof: If $s \le g \le S$ for some numbers $s, S \in \mathbb{R}$, then $s\tau \le \Delta_g \le S\tau$, and

$$P_\sigma((s - t)\tau) \le P_\sigma(\Delta_g - t\tau) \le P_\sigma((S - t)\tau).$$

Under the above assumptions there exist $t_s, t_S \in \mathbb{R}$ such that

$$0 \le P_\sigma((s - t_s)\tau) \le P_\sigma(\Delta_g - t_s\tau)$$

and

$$P_\sigma(\Delta_g - t_S\tau) \le P_\sigma((S - t_S)\tau) \le 0.$$

Again, the continuity of the pressure ensures the existence of a root. □

The next lemma shows that under the above assumptions the thermodynamic formalism is identical to the one for suspension flows over compact subshifts of finite type.

Proposition 8. *If $g : Y \to \mathbb{R}$ is bounded and Δ_g is locally Hölder, then the function $t \mapsto P_\Phi(tg)$ is real analytic.*

Proof: It follows from the proof of Proposition 7 that $P_\Phi(tg)$ is finite for every $t \in \mathbb{R}$. As in the proof of Proposition 6, the implicit function theorem yields the desired result. □

4.3. Extension of Potentials Defined on the Base

Let $\phi : \Sigma \to \mathbb{R}$ be locally Hölder. It is shown in Ref. 1 that there exists a continuous function $g : Y \to \mathbb{R}$ such that $\Delta_g = \phi$. This provides a tool to construct examples.

Namely, let $f : \Sigma \to \mathbb{R}$ be a locally Hölder function, bounded above, and with $P_\sigma(f) = 0$. Let $\varepsilon > 0$ and consider

$$\Sigma^+ = \{x \in \Sigma : f(x) > -\varepsilon\} \quad \text{and} \quad \Sigma^- = \{x \in \Sigma : f(x) \le -\varepsilon\}.$$

We define $\tau, \phi : \Sigma \to \mathbb{R}$ by

$$\tau(x) = \begin{cases} -f(x), & x \in \Sigma^- \\ \varepsilon, & x \in \Sigma^+ \end{cases} \quad \text{and} \quad \phi(x) = \begin{cases} 0, & x \in \Sigma^- \\ f(x) + \varepsilon, & x \in \Sigma^+. \end{cases}$$

Note that $\tau \ge \varepsilon$ and $\phi - \tau = f$. Then

$$P_\sigma(\Delta_g - \tau) = P_\sigma(\phi - \tau) = P_\sigma(f) = 0$$

Therefore, $P_\Phi(g) = 1$ and the recurrence properties of f can be transferred into recurrence properties of g. More precisely, using the language Refs. 15 and 16:

1. if f is positive recurrent and the corresponding conformal measure belongs to $\mathcal{M}_\sigma(-\tau)$, then g has a conservative conformal measure and there exists an equivalent invariant probability measure;

2. if f is positive recurrent and the corresponding conformal measure does not belong to $\mathcal{M}_\sigma(-\tau)$, then g has a conservative conformal measure and there exists an equivalent invariant infinite measure (the same holds if f is null recurrent);
3. if f is transient, then g has no conservative conformal measure.

5. MULTIFRACTAL ANALYSIS

In this section we study the multifractal analysis of suspension semiflows over countable Markov shifts. More precisely, we study the entropy spectra of Birkhoff averages. The case of suspension flows over finite Markov shifts was studied by Barreira and Saussol in Refs. 2 and 3 (see also Ref. 13).

5.1. Entropy of Arbitrary Sets

In the theory of multifractal analysis there are several ways to measure the "size" of a set. Here we consider the topological entropy. We note that the level sets $J_\alpha \subset Y$ of a multifractal decomposition (see (18) in Sec. 5.2) are not obtained from a Markov shift $\Sigma' \subset \Sigma$ as in (1), i.e., we cannot replace the pair (Y, Σ) by (J_α, Σ') in (1) (unless J_α is the whole space). This means that we need an appropriate notion of topological entropy in *arbitrary* subsets for suspension semiflows over countable Markov shifts. To the best of our knowledge, no such notion exists in the literature.

Let $Z \subset Y$ be an arbitrary set (not necessarily compact nor invariant). We define the *topological entropy* of Φ on Z by

$$h^*(\Phi|Z) := \sup\{h_B(\Phi|Z \cap K) : K \subset Y \text{ compact and } \Phi\text{-invariant}\}, \tag{16}$$

where $h_B(\Phi|Z \cap K)$ is Bowen's notion of topological entropy on an arbitrary subset of a compact invariant set (with respect to the Bowen-Walters distance on Y; see Sec. 2.3).

We show that $h^*(\Phi|Z)$ is an extension both of Bowen's notion of topological entropy (on noncompact sets) and of our notion in (7) (and thus we have the right to continue calling it topological entropy).

Proposition 9. *The following properties hold:*

1. *for any set $Z \subset K' \subset Y$, where K' is compact and Φ-invariant, we have $h^*(\Phi|Z) = h_B(\Phi|Z)$;*
2. *if $Z \subset Y$ is obtained from a Markov shift $\Sigma' \subset \Sigma$ as in (1), i.e.,*

$$Z = \{(x, t) \in \Sigma' \times \mathbb{R} : 0 \le t \le \tau(x)\}, \tag{17}$$

then $h^*(\Phi|Z) = h(\Phi|Z)$.

Proof: For the first property, note that since $Z = Z \cap K'$, it follows from the definitions that

$$h^*(\Phi|Z) = h^*(\Phi|Z \cap K') = h_B(\Phi|Z \cap K') = h_B(\phi|Z).$$

For the second property, note that by (7) and Theorem 3,

$$\begin{aligned} h(\Phi|Z) :&= \inf\{t \in \mathbb{R} : P_{\sigma|\Sigma'}(-t\tau) \leq 0\} \\ &= \sup\{h(\Phi|K) : K \subset Z \text{ compact and } \Phi\text{-invariant}\}. \end{aligned}$$

Since

$$h(\Phi|K) = h(\Phi|Z \cap K) = h_B(\Phi|Z \cap K),$$

we obtain $h(\Phi|Z) \leq h^*(\Phi|Z)$. On the other hand, by Theorem 3 and (17),

$$\begin{aligned} h_B(\Phi|Z \cap K) &= h(\Phi|Z \cap K) \\ &= \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}_\Phi \text{ with } \mu(Z \cap K) = 1\} \\ &\leq \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}_\Phi \text{ with } \mu(Z) = 1\} = h(\Phi|Z), \end{aligned}$$

and hence $h^*(\Phi|Z) \leq h(\Phi|Z)$. Therefore, $h^*(\Phi|Z) = h(\Phi|Z)$. □

In view of Proposition 9 we will denote from now on both topological entropies $h^*(\Phi|Z)$ and $h_B(\Phi|Z)$ by $h(\Phi|Z)$.

5.2. Entropy Spectra and Multifractal Analysis

We always assume in this section that the countable Markov shift in the base is topologically mixing and satisfies the BIP property. Recall that in this setting the pressure, when finite, is real analytic (see Sec. 2.2). We also assume that there exists $t_c > 0$ such that $P_\sigma(-t\tau) < \infty$ for every $t > t_c$, and that (15) holds.

We consider the multifractal decompositions induced by Birkhoff averages. Let $g : Y \to \mathbb{R}$ be a continuous function such that Δ_g is locally Hölder. Given $\alpha \in \mathbb{R}$, we consider the level set

$$J_\alpha := \left\{x \in Y : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi_r x)\, dr = \alpha\right\},$$

where $(\varphi_t)_{t\geq 0}$ is the suspension semiflow. We also consider the *irregular* set

$$J' := \left\{x \in Y : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi_r x)\, dr \text{ does not exist}\right\}.$$

The multifractal decomposition is given by the disjoint union

$$Y = \left(\bigcup_{\alpha\in\mathbb{R}} J_\alpha\right) \cup J'. \tag{18}$$

Note that by Birkhoff's ergodic theorem, $\mu(J') = 0$ for every $\mu \in \mathcal{M}_\Phi$.

The *entropy spectrum* of the Birkhoff averages of g is defined by

$$\mathcal{E}(\alpha) = h(\Phi|J_\alpha).$$

We say that two functions $g, h : Y \to \mathbb{R}$ are *cohomologous* if there exists a locally bounded measurable function $\rho : Y \to \mathbb{R}$ such that

$$g(x) - h(x) = \lim_{t \to 0^+} \frac{\rho(\varphi_t x) - \rho(x)}{t} \text{ for every } x \in Y.$$

The following is our main result on the entropy spectrum.

Theorem 10. *Let $g : Y \to \mathbb{R}$ be a continuous function noncohomologous to a constant and with $P_\Phi(g) = 0$, such that Δ_g is locally Holder and nonpositive. Then either:*

1. *$\mathcal{E}$ is real analytic, strictly concave, and its domain is a closed bounded interval;*
2. *$\mathcal{E}$ is real analytic, strictly concave, and its domain is unbounded;*
3. *$\mathcal{E}$ has unbounded domain, and there exists $\beta \in \mathbb{R}$ such that for $t \geq \beta$ the spectrum is strictly concave and for $t < \beta$ the spectrum is constant equal to $h(\Phi)$.*

In all cases the irregular set J' has full entropy, i.e., $h(\Phi|J') = h(\Phi)$.

Proof: It was proved by Barreira and Saussol in Ref. 2 (building on work of Barreira and Schmeling[(4)]) that if Φ is a suspension flow over a compact subshift of finite type and Δ_g is Hölder continuous, then the irregular set of a function g not cohomologous to a constant has full topological entropy. On the other hand, the cohomology assumption implies that there is an increasing sequence of compact Φ-invariant sets K_n with $\bigcup_{n \in \mathbb{N}} K_n = \Sigma$ such that $g|K_n$ is not cohomologous to a constant (up to a bounded measurable function). Theorem 1 implies that

$$\begin{aligned} h(\Phi) &= \sup\{h(\Phi|K) : K \text{ compact and } \Phi\text{-invariant}\} \\ &\geq \sup\{h(\Phi|J' \cap K) : K \text{ compact and } \Phi\text{-invariant}\} = h(\Phi|J') \\ &\geq \sup\{h(\Phi|J' \cap K) : K \text{ compact and } \sigma|K \text{ is Markov}\} = h(\Phi), \end{aligned}$$

where the second equality is due to our definition of entropy, and where the last one follows from the cited work in Ref.2. This shows that the irregular set has full topological entropy.

We now set $T(q) = P_\Phi(qg)$ for each $q \in \mathbb{R}$. Under our assumptions the function $T(q)$ can either be:

1. finite and analytic for $q \in \mathbb{R}$;

2. finite for $q \geq 0$ and infinite for $q < 0$, and either

$$\lim_{q \to 0^+} \frac{d}{dq} P(qg) = \infty \quad \text{or} \quad \lim_{q \to 0^+} \frac{d}{dq} P(qg) < \infty.$$

We first assume that T is finite and analytic. In this case the approach of Barreira and Saussol in Ref.2 can be used without change to show that $\mathcal{E}$ is real analytic and strictly concave. Furthermore, arguments of Schmeling in Ref.18 show that the domain of $\mathcal{E}$ is a closed bounded interval.

Assume from now on that $T(q) = \infty$ for negative values of q. It is shown in Ref.2 that if Φ is a suspension flow on a compact set K and Δ_g is Hölder continuous, then the domain of $\mathcal{E}$ is the range of the derivative function T', which is a closed bounded interval (we note that in the identity (19) in Ref.2 we must add a minus sign before the integral, and thus $\mathcal{E}(-T'(q))$ must also be replaced by $\mathcal{E}(T'(q))$ in Theorem 9 in Ref. 2).

To show that the domain of $\mathcal{E}$ is unbounded we use an approximation argument. Set $T_K(q) = P_{\Phi|k}(qg)$. Due to the approximation property in Theorem 1, we have

$$T(q) = \sup\{T_K(q) : K \subset Y \text{ compact and } \Phi\text{-invariant}\}.$$

We recall that for each compact and Φ-invariant set $K \subset Y$, the function T_K is real analytic, and the range of T'_K is a bounded interval.

Note that $T_K(0) \leq T(0) = h(\Phi) < \infty$ (our assumptions on the height function ensure that $h(\Phi) < \infty$). Take $a < 0$. Since $T(-1) = \infty$ there exists a compact and Φ-invariant set $K \subset Y$ such that $|T_K(-1) - h(\Phi)| > -a$. Since T_K is real analytic we can apply the mean value theorem, and there exists $p \in [-1, 0]$ such that

$$T_K(0) - T_K(-1) = T'_K(p).$$

Therefore the range of T'_K is a bounded interval $[s, S]$ (depending on K) with $s < a$. The fact that the domain of $\mathcal{E}$ is unbounded follows from the inclusion of each interval $[s, S]$ in the domain.

Set now

$$\beta = \lim_{q \to 0^+} \frac{d}{dq} P(qg). \tag{19}$$

When $\beta = \infty$, the range of T' is unbounded. It follows from work by Barreira and Saussol in Ref. 2 that

$$\mathcal{E}(T'(q)) = T(q) - qT'(q), \tag{20}$$

and the result follows immediately from this relation. When $\beta < \infty$ we have $\mathcal{E}(\beta) = T(0)$, which is an upper bound for $\mathcal{E}$. Since the domain is unbounded and the function $\mathcal{E}$ is concave, the spectrum satisfies $\mathcal{E}(t) = h(\Phi)$ for $t < \beta$. □

Recall that for suspension flows over finite Markov shifts the entropy spectrum is analytic and has bounded domain. This strongly contrasts with what happens here. In particular, the domain of the spectrum may be unbounded (see statements 2 and 3 of Theorem 10), and the spectrum may have points where it is not analytic (see statement 3). Recall that our assumptions on the height function ensure that $h(\Phi) < \infty$. Similar results can be obtained if we allow the flow to have infinite entropy (in which case the spectrum has unbounded image).

5.3. Bounded Versus Unbounded Domain

To give an example corresponding to statement 1 in Theorem 10 it suffices to consider a bounded potential $g : Y \to \mathbb{R}$. As noted in Proposition 8, the function T is then analytic in $\mathbb{R}$. Therefore the classical theory applies, and the entropy spectrum is real analytic, strictly concave, and has bounded domain.

We now turn to the case of unbounded domain. We first describe the basic idea to construct examples of this type. We write $g = \log f$, and we assume that $0 < f < 1$. The heuristic argument is as follows:

1. in order that

$$\left|\lim_{t\to\infty} \frac{1}{t}\int_0^t \log f(\varphi_r x)dr\right|$$

is arbitrarily large for certain points x, we want f to be sufficiently close to zero; this means that for $q < 0$ the potential $q \log f$ should be sufficiently "large";
2. the notion of "large" is related to τ: we need that $q \log f$ is "larger" than τ, in the sense that $P(q \log f) = \infty$ for $q < 0$; this means that

$$\inf\{t \in \mathbb{R} : P_\sigma(\Delta_{q\log f} - t\tau) \le 0\} = \infty;$$

3. in conclusion, if $\Delta_{-\log f}$ is much "larger" than τ, then the spectrum has unbounded domain.

Example 3. Let σ be the full shift defined on $\mathbb{N}$, and consider the height function $\tau(x) := \log(x_0(x_0 + 1))$. We know from Example 1 that $h(\Phi) = 1$. Consider now the locally constant potential defined by $\phi(x) = -x_0 \log x_0$, and let $g : Y \to \mathbb{R}$ be a continuous function such that $\Delta_g = \phi$. It follows from (9) that for $q < 0$,

$$\begin{aligned} P_\Phi(qg) &= \inf\{t \in \mathbb{R} : P_\sigma(q\Delta_g - t\tau) \le 0\} \\ &\le \inf\{t \in \mathbb{R} : P_\sigma(q\phi - t\tau) \le 0\} \\ &= \inf\left\{t \in \mathbb{R} : \log\sum_{n=1}^{\infty} \frac{n^{qn}}{(n(n+1))^t} \le 0\right\} = \infty. \end{aligned}$$

Hence, the entropy spectrum has unbounded domain.

5.4. Examples for Statements 2 and 3 in Theorem 10

We assume here that the entropy spectrum has unbounded domain. Given a function $g : Y \to \mathbb{R}$, we are assuming that $P_\Phi(qg) = \infty$ for every $q < 0$. Note that, since the entropy is finite, we have $P_\Phi(0) < \infty$. In statements 2 and 3 of Theorem 10 the shape of the spectrum depends on β in (19).

Lemma 1. *The function T is analytic in (0,1).*

Proof: Let $q \in (0, 1)$. Since $\Delta_g \le 0$ we have that $\Delta_g \le q\Delta_g$. Therefore

$$P_\sigma(\Delta_g - t\tau) \le P_\sigma(q\Delta_g - t\tau) \le P_\sigma(-t\tau).$$

Thus, there exist $t_s, t_S \in \mathbb{R}$ such that

$$P_\sigma(q\Delta_g - t_s\tau) \le 0 \quad \text{and} \quad 0 \le P_\sigma(q\Delta_g - t_S\tau) < \infty.$$

The continuity of the pressure ensures the existence of a root of the equation $P_\sigma(q\Delta_g - t\tau) = 0$ for every $q \in (0, 1)$. Since the shift σ has the BIP property, the analyticity of T follows from the implicit function theorem. □

The derivative of the pressure is given by

$$\frac{d}{dq} P_\sigma(qg)|_{q=p} = \int_Y g d\mu_p = \frac{\int_\Sigma \Delta_g d\nu_p}{\int_\Sigma \tau d\nu_p}, \tag{21}$$

where μ_p is the equilibrium measure for pg, and ν_p is the equilibrium measure for $p\Delta_g - P_\Phi(pg)\tau$. Denote by μ_0 the measure of maximal entropy for the flow (note that it exists since Σ is the full shift and $P(-h(\Phi)\tau) = 0$), and by υ_0 the equilibrium (Gibbs) measure for $-h(\Phi)\tau$ (see Ref.16). We have

$$T'(0) = \frac{\int_\Sigma \Delta_g d\nu_0}{\int_\Sigma \tau d\nu_0}$$

Since τ is bounded away from zero, it follows from the variational principle that

$$0 < \int_\Sigma \tau d\nu_0 = \frac{h_{\nu_0}(\sigma)}{h(\Phi)} < \infty$$

The problem, depending on whether $\beta < \infty$ or $\beta = \infty$, is thus reduced to the integrability of Δ_g with respect to ν_0, and we can obtain general assumptions for each situation in statements 2 and 3 of Theorem 10.

Proposition 11. *Let ν_0 the equilibrium measure for $-h(\Phi)\tau$.*

1. *If $\Delta_g \notin L^1_{\nu_0}$ then the spectrum is analytic and strictly concave.*
2. *If $\Delta_g \in L^1_{\nu_0}$ then the spectrum is analytic and strictly concave up to some critical point after which the spectrum is constant.*

This criterion can be used to construct examples.

Example 4. $(-T'(0) = \infty)$. With the same setting as in Example 3, we consider the locally constant function defined by $\phi(x) = -(x_0)^2 \log x_0$. Let $g : Y \to \mathbb{R}$ be a continuous function such that $\Delta_g = \phi$. Since ν_0 is a Gibbs measure we have

$$-\int_\Sigma \Delta_g \, d\nu_0 = -\sum_{n=1}^{\infty} \int_{C_n} \Delta_g d\nu_0 = \sum_{n=1}^{\infty} -\Delta_{g|C_n} \nu_0(C_n)$$
$$\geq K \sum_{n=1}^{\infty} \frac{n^2 \log n}{n(n+1)} = \infty$$

for some constant $K > 0$. In this case the spectrum has unbounded domain, is analytic, and is strictly concave.

Example 5. $(-T'(0) < \infty)$. Let σ be the full shift on $\mathbb{N}$, and $\tau(x) := x_0 \log 2$. Using (9) we obtain

$$P_\sigma(-\tau) = \log \sum_{n=1}^{\infty} 2^{-n} = 0.$$

Hence, $h(\Phi) = 1$. Define $\phi(x) = \log(x_0(x_0 + 1))$ and let $g : Y \to \mathbb{R}$ be a continuous function such that $\Delta_g = \phi$. Since ν_0 is a Gibbs measure we have

$$-\int_\Sigma \Delta_g d\nu_0 = \sum_{n=1}^{\infty} -\Delta_g|_{C_n} \nu_0(C_n) \leq K \sum_{n=1}^{\infty} \frac{\log(n(n+1))}{2^n} < \infty$$

for some constant $K > 0$. In this case the spectrum has unbounded domain, and there exists $\beta \in \mathbb{R}$ such that

$$\mathcal{E}(\alpha) = \begin{cases} \text{strictly concave} & \text{if } \alpha > \beta \\ 1 & \text{if } \alpha \leq \beta \end{cases}.$$

5.5. Existence of Full Measures

A Φ-invariant probability measure μ_α is called a *full* measure for the level set J_α if $h_{\mu_\alpha}(\Phi) = h(\Phi|J_\alpha)$. The existence of full measures is closely related to the existence of equilibrium measures for $-T(q) + qg$, for an appropriate range of values of the parameter q.

The problem can be reduced to a corresponding problem for the Markov shift. By Theorem 4, since $P_\Phi(-T(q) + qg) = 0$, there is an equilibrium measure for $-T(q) + qg$ if and only if

$$P_\sigma\left(\Delta_{-T(q)+qg}\right) = 0$$

and there is an equilibrium measure for $\Delta_{-T(q)+qg}$. The advantage of the reduction is that since the Markov shift σ satisfies the BIP property, the potential $q\Delta_g - T(q)\tau = \Delta_{-T(q)+qg}$ has a unique Gibbs measure ν_q. In the case of finite Markov shifts, a *full* measure is obtained from the product of ν_q and Lebesgue measure. The difficulty in the present setting is that the Gibbs measure might not be an equilibrium measure. In fact, it can happen that

$$h_{\nu_q}(\sigma) = \infty \quad \text{and} \quad \int_\Sigma (q\Delta_g - T(q)\tau) d\nu_q = -\infty,$$

and thus the sum of the two terms is meaningless. Note that we are assuming that $h(\Phi) < \infty$, and thus $h_{\nu_q}(\sigma) = \infty$ implies that $\int_\Sigma \tau d\nu_q = \infty$, that is, $\nu_q \notin \mathcal{M}_\sigma(-\tau)$ (see Example 6 below). Therefore, $\nu_q \in \mathcal{M}_\sigma(-\tau)$ if and only if the measure μ_q obtained from the product $\nu_q \times m$ is an equilibrium measure for $-T(q) + qg$.

Theorem 12. *Under the hypotheses of Theorem 10, the following holds:*

1. *if $T(q) = \infty$ for negative values of q and $\beta < \infty$, then for $\alpha < \beta$ there is no full measure for J_α;*
2. *if $q \in \mathbb{R}$ is such that $T'(q) = \alpha$ and $\nu_q \in \mathcal{M}_\sigma(-\tau)$, then there is a full measure for J_α when:*
   (a) *T is real analytic in $\mathbb{R}$ and $\alpha \in (\alpha_{\min}, \alpha_{\max})$;*
   (b) *$T(q) = \infty$ for negative values of q, $\beta = \infty$, and $\alpha \in (-\infty, \alpha_{\max})$;*
   (c) *$T(q) = \infty$ for negative values of q, $\beta < \infty$, and $\alpha \in (-\beta, \alpha_{\max})$.*

Proof: To prove statement 1 assume on the contrary that there is a full measure μ_α for J_α, where $\alpha < \beta$. Recall that for $\alpha < \beta$ the level set has full topological entropy, $h(J_\alpha) = h(\Phi)$. Therefore, if

$$h_{\mu_\alpha}(\sigma) = h(\Phi|J_\alpha) = h(\Phi),$$

then μ_α is a measure of maximal entropy. But μ_β is also a measure of maximal entropy and there is at most one (there is at most one equilibrium measure for $-h(\Phi)\tau$; see [Theorem 1.1] in Ref. 8).

For statement 2, let $\alpha(q) = T'(q)$. Denote by μ_q the unique equilibrium measure for $-T(q) + qg$. By (21), we have

$$\int_Y g\, d\mu_q = \alpha(q) \quad \text{and} \quad \mu_q(J_{\alpha(q)}) = 1.$$

Moreover, $\mu_q = R(\nu_q)$ where ν_q is the unique equilibrium (Gibbs) measure for $q\Delta_g - T(q)\tau$. In fact

$$\sup\left\{h_\nu(\sigma) + \int_\Sigma (q\Delta_g - T(q)\tau)d\nu : \nu \in \mathcal{M}_\sigma\right\}$$
$$= h_{\nu_q}(\sigma) + \int_\Sigma (q\Delta_g - T(q)\tau)d\nu_q = P_\sigma(q\Delta_g - T(q)\tau) = 0.$$

We obtain

$$\frac{h_{\nu_q}(\sigma)}{\int_\Sigma \tau d\nu_q} = T(q) + q\frac{\int_\Sigma \Delta_g d\nu_q}{\int_\Sigma \tau d\nu_q}.$$

Therefore, by Abramov's formula and (20),

$$h_{\mu_q}(\Phi) = T(q) + q\int_Y g d\mu_q = T(q) + q\alpha(q) = \mathcal{E}(\alpha(q)).$$

The measure μ_q is the unique full measure for $J_{\alpha(q)}$. Otherwise, if $\mu \in \mathcal{M}_\Phi$ is such that $\mu \neq \mu_q$, $\mu(J_{\alpha(q)}) = 1$, and

$$h_\mu(\Phi) = T(q) - q\alpha(q),$$

then by the variational principle and the uniqueness of the equilibrium measure we obtain

$$h_\mu(\Phi) + q\alpha(q) = T(q) = P_\Phi(qg)$$
$$= h_{\mu_q}(\Phi) + q\int_Y g d\mu_q > h_\mu(\Phi) + q\int_Y g d\mu.$$

Therefore $\alpha(q) > \int_Y g d\mu$. On the other hand, since $\mu(J_{\alpha(q)}) = 1$ we have that $\alpha(q) = \int_Y g d\mu$. This contradiction completes the proof. □

We now give an example of a suspension flow and of a function g such that $P_\sigma(\Delta_g - P_\Phi(g)\tau) = 0$ and there exists an equilibrium measure $\nu \in \mathcal{M}_\sigma$ for $\Delta_g - P_\Phi(g)\tau$ but $\int_\Sigma \tau d\nu = \infty$. Therefore, g has no equilibrium measure.

Example 6. Consider a full shift defined on $\mathbb{N}$, and the height function τ given on cylinders by $\tau|C_k = k\log k$. Let also $f : \Sigma \to \mathbb{R}$ be defined by

$$f|C_k := -\log(k(k+1)),$$

and consider a function $g : Y \to \mathbb{R}$ such that $\Delta_g = f - \tau$. We obtain

$$P_\sigma(\Delta_g + \tau) = P_\sigma(f - \tau + \tau) = 0.$$

Therefore, $P_\Phi(g) = -1$. Moreover, since the system satisfies the BIP property, there exists a Gibbs measure ν corresponding to $\Delta_g + \tau$. Nevertheless,

$$\int_\Sigma \tau d\nu = \sum_{k=1}^{\infty} \tau|_{C_k} \nu(C_k) = \sum_{k=1}^{\infty} \frac{k \log k}{k(k+1)} = \infty.$$

ACKNOWLEDGMENTS

The authors would like to thank Claudia Valls for her help. The authors also would like to thank the referees for the careful reading of their manuscript and for many suggestions. This work was supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems, and through Fundação para a Ciência e a Tecnologia by Program POCTI/FEDER, and the grant SFRH/BPD/21927/2005.

REFERENCES

1. L. Barreira, L. Radu and C. Wolf, Dimension of measures for suspension flows. *Dyn. Syst.* **19**: 89–107 (2004).
2. L. Barreira and B. Saussol, Multifractal analysis of hyperbolic flows. *Comm. Math. Phys.* **214**: 339–371 (2000).
3. L. Barreira and B. Saussol, Variational principles for hyperbolic flows, in Differential Equations and Dynamical Systems (Lisboa, 2000), Eds. A. Galves, J. Hale and C. Rocha, Fields Inst. Comm. 31. *Amer. Math. Soc.* pp. 43–63 (2002).
4. L. Barreira and J. Schmeling, Sets of “non-typical” points have full topological entropy and full Hausdorff dimension. *Israel J. Math.* **116**: 29–70 (2000).
5. R. Bowen, Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95**: 429–460 (1973).
6. R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.* **29**: 181–202 (1975).
7. R. Bowen and P. Walters, Expansive one-parameter flows. *J. Differential Equations* **12**: 180–193 (1972).
8. J. Buzzi and O. Sarig, Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. *Ergodic Theory Dynam. Systems* **23**: 1383–1400 (2003).
9. B. Gurevich and S. Katok, Arithmetic coding and entropy for the positive geodesic flow on the modular surface. *Mosc. Math. J.* **1**: 569–582 (2001).
10. G. Iommi, Multifractal analysis for countable Markov shifts. *Ergodic Theory Dynam. Systems*, **25**: 1881–1907 (2005).
11. R. Mauldin and M. Urbański, Gibbs states on the symbolic space over an infinite alphabet. *Israel J. Math.* **125**: 93–130 (2001).
12. W. Parry and M. Pollicott, Zeta Functions and the Periodic Orbit Structure of Hyperbolics Dynamics. *Astérisque* 187–188 (1990).
13. Ya. Pesin and V. Sadovskaya, Multifractal analysis of conformal axiom A flows. *Comm. Math. Phys.* **216**: 277–312 (2001).

14. M. Ratner, Markov partitions for Anosov flows on n-dimensional manifolds. *Israel J. Math.* **15**: 92–114 (1973).
15. O. Sarig, Thermodynamic formalism for countable Markov shifts. *Ergodic Theory Dynam. Systems* **19**: 1565–1593 (1999).
16. O. Sarig, Existence of Gibbs measures for countable Markov shifts. *Proc. Amer. Math. Soc.* **131**: 1751–1758 (2003).
17. S. Savchenko, Special flows constructed from countable topological Markov chains. *Funct. Anal. Appl.* **32**: 32–41 (1998).
18. J. Schmeling, On the completeness of multifractal spectra. *Ergodic Theory Dynam. Systems* **19**: 1595–1616 (1999).